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THE DIRAC HAMILTONIAN FORMALISM AND THE REALIZATION OF CONSTRAINTS BY SMALL MASSES†

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The Dirac problem of the Hamiltonian formalism of systems with constraints and the realization of constraints by small masses [1, 2] is considered. It is shown that, with a mass tending towards zero under certain initial conditions, limiting motions exist and match the motions of the Hamiltonian system with constraints. The results obtained are used in the problem of realizing a unilateral holonomic constraint. © 2000 Elsevier Science Ltd. All rights reserved.

It is well known that, to describe the motion of Hamiltonian systems with constraints, it is possible to use the so-called "generalized Dirac Hamiltonian formalism" [1, 2]. The Dirac problem reduces to an investigation of the variational Lagrange problem [2, 3], and here the Lagrange function is singular with respect to velocities. It has been shown [2] that a constraint can be realized by small masses, and solutions of the singularly perturbed equations obtained can be sought in the form of formal expansions in series in powers of a small parameter.

Below it is shown that, when a mass tends to zero, limiting motions exist (for an appropriate choice of the initial conditions), and the formal series indicated are asymptotic. These results are applied to the problem of realizing a unilateral holonomic constraint by elastic forces.

1. FORMULATION OF THE PROBLEM

Following the well-known approach [2], we shall examine a Hamiltonian system on a manifold M of dimension $2n$ with Hamiltonian

$$H = H_0(p,q,Q) + P^2/(2\varepsilon) + \varepsilon H_1(p,q,Q,\varepsilon) \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, and the function H_0 is not degenerate with respect to the momenta p. Here $\{p, q\} \in R^{2n-2}$. We shall assume that all the functions are smooth.

The constraint is set by the equality $P = 0$, and in this case the compatibility condition

$$\{P, H_0\} = -H_{0Q} = 0 \tag{1.2}$$

must be satisfied.

Let $Q = f(p, q)$ be the solution of Eqs (1.2). It was shown in [2] that the Hamiltonians of the Dirac equation with a constraint take the form

$$\dot{p} = -H^*_{0q}, \quad \dot{q} = H^*_{0p}, \quad P = 0, \quad Q = f \tag{1.3}$$

where $H^*_0(p, q) = H_0(p, q, Q)|_{Q=f}$. We shall assume that

$$H_{0QQ}(p,q,Q)\big|_{Q=f} > 0 \tag{1.4}$$

Here and below, the subscripts Q, q, p, etc. denote partial derivatives with respect to the corresponding variables.

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2. A THEOREM ON TAKING THE LIMIT

Let p_0 and q_0 – the solutions of Eqs (1.3) – exist in the time interval [0, T]. Let p, q, P and Q be the solution of Hamilton's equations with Hamiltonian (1.1). We shall assume that, at the initial instant of time, the following relations are satisfied

$$|p(0)-p_0(0)|+|q(0)-q_0(0)|<\sqrt{\varepsilon},\quad |Q(0)-f(p_0(0),q_0(0))|<\sqrt{\varepsilon},\quad |P(0)|<\varepsilon \tag{2.1}$$

Theorem. For sufficiently small ε in the interval [0, T], the following estimates hold

$$Q-f(p_0,q_0)=O(\sqrt{\varepsilon}),\quad P=O(\varepsilon),\quad |p-p_0|=O(\sqrt{\varepsilon}),\quad |q-q_0|=O(\sqrt{\varepsilon})$$

Proof. It is convenient to expand the function H_0 of Hamiltonian (1.1) in a series in powers of $(Q-f)$

$$H_0=H_0^*(p,q)+\frac{1}{2}H_{0QQ}\big|_{Q=f}(Q-f(p,q))^2+O(Q-f)^3+\dots$$

$$H_0^*(p,q)=H_0(p,q,f(p,q))$$

Note that $H_0Q|_{Q=f}=0$, since $Q=f(p,q)$ is the solution of Eq. (1.2).
We make a replacement of variables with the generating function

$$S=p_1q+P_1(Q-f(p_1,q))$$

In the new variables, the equations of motion will take the form

$$\begin{aligned}
\dot p_1&=-H_{0q}^*(p_1,q_1)+O(P_1)+O(\varepsilon),\quad \dot q=H_{0p}^*+O(P_1)+O(\varepsilon)\\
\dot P_1&=-H_{0QQ}Q_1+Q_1^2F_1(p_1,q_1,Q_1,\varepsilon)+P_1G_1(p_1,q_1,P_1,Q_1,\varepsilon)+O(\varepsilon)\\
\dot Q_1&=P_1/\varepsilon+g(p_1,q_1)+Q_1F_2(p_1,q_1,Q_1,\varepsilon)+P_1G_2(p_1,q_1,P_1,Q_1,\varepsilon)+O(\varepsilon)
\end{aligned} \tag{2.2}$$

where F, G and g are smooth functions which are determined from the replacement of variables.
We put $P^{(1)}=P_1/\sqrt{\varepsilon}$ and make the time replacement $\tau=t/\sqrt{\varepsilon}$:

$$\begin{aligned}
&p_1'=O(\sqrt{\varepsilon}),\quad q_1'=O(\sqrt{\varepsilon})\\
&P^{(1)\prime}=-H_{0QQ}Q_1+Q_1^2F_1+O(\sqrt{\varepsilon}),\quad Q_1'=P^{(1)}+O(\sqrt{\varepsilon})
\end{aligned} \tag{2.3}$$

The prime denotes a derivative with respect to τ.
System (2.3) is a classical system with fast and slow variables, and $\sqrt{\varepsilon}$ acts as a small parameter. Owing to inequality (1.4), the zeroth solution of the unperturbed system is stable. We can use the general results on the evolution of perturbed Hamiltonian systems [4]: a constant K exists such that, in the interval $0\leqslant\tau\leqslant T/\sqrt{\varepsilon}$, the following inequality is satisfied

$$|P_1/\sqrt{\varepsilon}|+|Q_1|+|p_1-p_0|+|q_1-q_0|<K\sqrt{\varepsilon}$$

which confirms the theorem.

3. THE SEARCH FOR A SOLUTION IN THE FORM OF A SERIES IN POWERS OF THE SMALL PARAMETER

Quantities of the order of unity on the right-hand side of the last equation of system (2.2) are the sum of two terms, one of which vanishes when P_1, Q_1, = 0. We shall seek a replacement of variables that eliminates the free term on the right-hand sides of the final two equations of system (2.2). We begin with the free term $g(p, q)$ of the order of unity

$$S = p_2 q_1 + (P_2 - \varepsilon g(p_2, q_1))Q_1 \tag{3.1}$$

In the new variables, Eqs (2.2) take the form

$$\begin{aligned} \dot{p}_2 &= -H^*_{0q}(p_2, q_2) + O(P_2) + O(\varepsilon), \quad \dot{q}_2 = H^*_{0p}(p_2, q_2) + O(P) + O(\varepsilon) \\ \dot{P}_2 &= -H_{0QQ}Q_2 + Q_2^2\hat{F} + O(P_2) + O(\varepsilon), \quad \dot{Q}_2 = P_2/\varepsilon + O(Q_2) + O(P_2) + O(\varepsilon) \end{aligned} \tag{3.2}$$

Here, $\hat{F}$, as in Section 2, is a smooth function of the variables p_2, q_2, and Q_2, and of the parameter ε. Now, the free term in the last two equations is a quantity of the order of ε.

The same replacements will be made as in the proof of the theorem. Then system (3.2) takes the form

$$\begin{aligned} p_2' &= O(\sqrt{\varepsilon}), \quad q_2' = O(\sqrt{\varepsilon}), \quad P^{(2)\prime} = -H_{0QQ}Q_2 + Q_2^2\hat{F} + \sqrt{\varepsilon}O(P^{(2)}) + O(\varepsilon^{3/2}) \\ Q_2' &= P^{(2)} + \sqrt{\varepsilon}O(Q_2) + \varepsilon O(P^{(2)}) + O(\varepsilon^{3/2}) \end{aligned} \tag{3.3}$$

We will assume that the initial conditions satisfy equalities (2.1), and therefore the variables p and q are bounded, while P and Q are small. The following estimates (cf. [4]) will be made: we introduce the function

$$Z = \frac{1}{2}(P^{(2)2} + H_{0QQ}Q_2^2) - \int_0^{Q_2} \tilde{Q}^2\hat{F}(\tilde{Q})d\tilde{Q}$$

By virtue of what has been said in Section 2, $Z = O(\varepsilon)$ in times of the order of $1/\sqrt{\varepsilon}$.

It can be verified that, for an arbitrary function Z with respect to the time τ, by virtue of system (3.3), the following estimate holds

$$Z' \leqslant M\sqrt{\varepsilon}Z + N\varepsilon^{3/2}\sqrt{Z} \tag{3.4}$$

where M and N are positive constants. In fact, since $Z = O(\varepsilon)$, we have $Z^{1+\alpha} \leqslant Z$ and $\alpha \geqslant 0$, and consequently all higher powers of P and Q are majorized by the function Z (multiplied, perhaps, by some constant). Solving inequality (3.4), we obtain

$$\int_0^{T/\sqrt{\varepsilon}} \frac{dZ}{MZ + N\varepsilon\sqrt{Z}} \leqslant \sqrt{\varepsilon}\int_0^{T/\sqrt{\varepsilon}} d\tau$$

$$M\sqrt{Z\left(\frac{T}{\sqrt{\varepsilon}}\right)} + N\varepsilon \leqslant \left(M\sqrt{Z(0)} + N\varepsilon\right)\exp\frac{T}{2}$$

Therefore, if at the initial instant the function Z was of the order of ε^2, then, in the entire time interval $\tau \in [0, T/\sqrt{\varepsilon}]$, it will be of the same order. Returning to the variables $P^{(2)}$ and Q_2, we establish that they are quantities of the order of ε. This means that, for the initial system (1.1), $P = -\varepsilon g(p, q) + O(\varepsilon^{3/2})$, $Q = f(p, q) + O(\varepsilon)$ is the solution. However, these terms are the first in the power series [2], which can be verified by direct substitution.

Then, making replacements similar to replacement (3.1), the free terms are eliminated successively. Let the free term of order ε in the penultimate equation of system (3.2) be equal to $\varepsilon g_1(p, q)$. Replacement with the generating function

$$S = p_3 q_2 + P_3(Q_2 - \varepsilon g_1(p_3, q_2)/H_{0QQ}(p_3, q_2))$$

eliminates the free term of the order of ε in the penultimate equation of system (3.2). In the final equation of (3.2), the free term, as before, is of the order of ε. Suppose it is equal to $\varepsilon g_2(p, q)$; then it is eliminated by the replacement

$$S = p_4 q_3 + \left(P_4 - \varepsilon^2 g_2(p_4, q_3)\right)Q_3$$

and so on. In this case we will obtain estimates similar to (3.4), and only power of ε in the second term of (3.4) will increase: at the kth step this power will be equal to $(2k + 1)/2$, and consequently $P^{(k)}$ and Q_k are quantities of the order of ε^k. In the initial variables, we obtain the required power series.

Remark. As a rule, the series obtained is divergent since in the general case it is not possible to eliminate the free term [5]. When all the functions are analytical, the coordinates of the free terms may be replaced by a quantity of the order of $(-1/\sqrt{\varepsilon})$. It must be pointed out that, in the procedure described above, in order to eliminate the next free term of the order of ε^k, we make two successive replacements (cf. [5]).

4. REALIZATION OF A UNILATERAL CONSTRAINT

Consider the problem of realizing a unilateral holonomic constraint by elastic forces with a large coefficient of elasticity [6, 7]. Suppose the "free" system is specified by the Hamiltonian

$$H = H_0(p,q,Q) + a(q,Q)P^2/2 + NV(Q) \tag{4.1}$$

$$V(Q) = \begin{cases} Q^2/2, & Q \leq 0 \\ 0, & Q > 0 \end{cases}, \quad N \gg 1$$

where p, P, q and Q are semigeodesic coordinates in which the unilateral constraint is specified by the condition $Q \geq 0$, while the quadratic form of the kinetic energy contains no products of p and P. It is well-known [8] that such coordinates always exist locally. For simplicity, we shall assume that they are introduced globally (cf. [9, 10]).

The equations of motion have the form

$$\begin{aligned} &\dot{p} = -H_{0q} - a_q P^2/2, \quad \dot{q} = H_{0p}, \quad \dot{Q} = aP \\ &\dot{P} = -H_{0Q} - a_Q P^2/2 - NQ, \quad Q \leq 0; \quad \dot{P} = -H_{0Q} - a_Q P^2/2, \quad Q > 0. \end{aligned} \tag{4.2}$$

Suppose that, in the time interval $[0, T]$, a system with a unilateral constraint moves on the constraint, and here, with $t \in [0, T)$, the constraint reaction is positive, while, at the instant $t = T$, leaving the constraint occurs and here the reaction of the corresponding system with a bilateral constraint has a simple zero [7, 10].

The solution of system (4.2) in the time interval $[0, T]$ generally fluctuates with amplitude $1/N$ [7]. When investigations an actual system it is necessary to solve the equations of motion in the half-spaces $Q < 0$ and $Q > 0$ and then to join the solutions. We shall show that, with the appropriate selection of the initial conditions of "free" system (4.2) from $(1/N)$ – the vicinity of the initial conditions of the corresponding system with a unilateral constraint in any time interval $[0, T_1] \in [0, T)$, motion occurs in the region $Q < 0$, and the amplitude of the oscillations in this case is a quantity $O(N^{-3/2})$ for fairly large values of the parameter N.

We will first consider the realization of a bilateral constraint, i.e. we will assume that the potential energy of the elastic force $V(Q)$ is equal to $Q^2/2$ in all cases. When $t \in [0, T)$, the reaction is positive, and consequently $H_{0Q} > 0$.

Let us assume that $\varepsilon = 1/N$ and make a canonical replacement of variables with the generating function

$$S = \hat{p}q + \hat{P}(Q + \varepsilon H_{0Q}(\hat{p}, q, Q))$$

In the new variables, the equations of motion will take a form similar to that of (2.2), apart from the replacement of P by Q, and therefore it is possible to use a similar method to that described in Section 3 (cf. [9]). In particular, we establish that, with an appropriate choice of the initial conditions

$$Q = -\varepsilon H_{0Q} + O(\varepsilon^{3/2})$$

and the conjugate momentum will be a quantity of the order of ε.

Remark. It is clear that the estimates obtained here do not contradict the theorem [9] in which the initial conditions for P and Q, generally speaking, could be any from the ε-vicinity of zero. The "revised" initial conditions physically correspond to placing a point on an elastic surface, waiting while the surface sags and only then pushing the point.

The coordinate Q vanishes when $H_{0Q} = O(\sqrt{\varepsilon})$. Thus, we have established that, in the time interval $[0, T + O(\sqrt{\varepsilon})]$, motion occurs in the half-space $Q < 0$. It must be noted that here quantities of the order of $\sqrt{\varepsilon}$ may be both positive and negative.

Let us return to the problem of realizing a unilateral constraint. Suppose the reaction is positive at the initial instant of time, i.e. $H_{0Q} > 0$. For any prescribed $0 < T_1 < T$ an ε exists such that $[0, T_1] \in [0, T + O(\sqrt{\varepsilon})]$. However, in the interval $[0, T + O(\sqrt{\varepsilon})]$ the motion of "free" system (4.2) will again occur in the half-space $Q < 0$, and the amplitude of the fluctuations about the "equilibrium" $Q = -\varepsilon H_{0Q}$ will be no more than $O(\varepsilon^{3/2})$. Then, in a time interval of the order of $\sqrt{\varepsilon}$, the quantity Q may oscillate and then becomes positive.

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